Complex dynamics in simple systems with periodic parameter oscillations

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We study systems with periodically oscillating parameters that can give way to complex periodic or nonperiodic orbits. Performing the long time limit, we can define ergodic averages such as Lyapunov exponents, where a negative maximal Lyapunov exponent corresponds to a stable periodic orbit. By this, extremely complicated periodic orbits composed of contracting and expanding phases appear in a natural way. Employing the technique of ϵ-uncertain points, we find that values of the control parameters supporting such periodic motion are densely embedded in a set of values for which the motion is chaotic. When a tiny amount of noise is coupled to the system, dynamics with positive and with negative nontrivial Lyapunov exponents are indistinguishable. We discuss two physical systems, an oscillatory flow inside a duct and a dripping faucet with variable water supply, where such a mechanism seems to be responsible for a complicated alternation of laminar and turbulent phases.

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I. INTRODUCTION

When studying deterministic dynamical systems, it has become practice to distinguish between chaotic and (quasi) periodic solutions, where chaos has been seen as novel and strange behavior. We study a class of dynamical systems, where such a distinction is not useful. Depending on control parameters, our systems will either generate stable periodic orbits or they will behave chaotically, but these two types of motion will be essentially indistinguishable from each other both on computers (due to finite precision) and in real experiments (due to weak noises).

Seasonal variations are a prominent source of slow periodic parameter fluctuations in biological, ecological, geochemical, and geophysical systems. But also in many technical and physical situations, slow periodic oscillations of system parameters do occur. Speaking of time-dependent parameters, we imply that there is no feedback from the system under study to the variation of the parameters, whose time dependence can either be considered as given (nonautonomous situation) or can be ruled by its own periodic autonomous dynamics. Moreover, we focus on situations where the typical time scales related to the system dynamics for fixed parameters are much faster than the time scale related to the parameter variation, as it is typical of many processes subject to seasonal variations. The opposite case, where both time scales are comparable, has been studied as an open-loop control mechanism [1,2].

In what follows, we discuss a scenario where slow harmonic parameter variations introduce an alternation of expanding and contracting phases. We will show that ergodic averages can be performed as usual and hence the motion in the long time limit is clearly classified to be either periodic or chaotic. However, as our analysis will show, in practical applications, chaotic motion will be indistinguishable from periodic motion, both in numerical simulations and in experiments. In the latter case, this is another example of "stable chaos" [3]. This indistinguishablility implies a robustness of the phenomenon despite the fact that there are parameter regimes where stable periodic motion and chaos are both supported by a dense set of parameters.

This rather unexpected behavior is shown to exist in numerical experiments of an oscillating flow inside a duct and a dripping faucet with variable liquid supply.

II. MAP WITH PERIODIC PARAMETER VARIATIONS

Many physical experiments and the corresponding model systems possess solutions which, due to dissipation, relax to rather uninteresting fixed points. Such systems are often exposed to a periodic driving, such as electric resonance circuits. In particular, when the system without driving has a two-dimensional phase space, chaos can only appear with the driving term.

This paper deals with a very different class of driven systems; we assume that our system without driving can behave chaotically, depending on the values of a control parameter. This parameter is then varied periodically, with a period that is much longer than the time scale of the internal dynamics on which the autocorrelation function in the chaotic regime would relax, or which would govern the relaxation to a stable fixed point in parameter regimes where it exists. Hence, the temporal dependence of the control parameter has no influence on the short time dynamics of the system, but it causes transitions between different dynamical behaviors which might exist for different values of the control parameter.

In what follows, we choose the quartic map f given by

$$x_{n+1} = f(a,x_n) = 4ax_n(1-x_n)[1-ax_n(1-x_n)] \qquad (1)$$

with $0 < a \leqslant 4$. This map has the interesting property that for $a = a_c = 3.375$, a tangent bifurcation occurs where out of a chaotic region that covers the whole [0, 1] interval a period-1 window appears. However, other maps can be also be con-

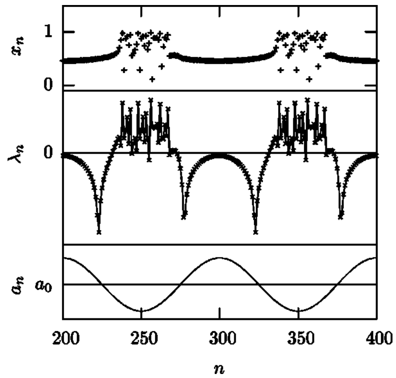

FIG. 1. Two hundred iterates of the map Eq. (2) for $a_0=3.4$, $a_1=0.1$, $\omega=2\pi/100$ (top panel), the instantaneous stretching factors λ_n (middle panel), and the parameter a_n (bottom panel).

sidered, for example the logistic map for values of the control parameter near a period-3 window.

Imposing a temporal dependence on the control parameter leads us to consider the two-dimensional skew autonomous system,

$$\phi_{n+1} = \phi_n + \omega,$$

$$x_{n+1} = f(a(\phi_n),x_n) = 4a(\phi_n)x_n(1-x_n)[1-a(\phi_n)x_n(1-x_n)], \tag{2}$$

where $a(\phi_n)=a_0+a_1 \cos \phi_n$, a_0 is chosen near a_c, $a_1 \ll 1$, and $\omega=2\pi/N$, $N \gg 1$. A small value of ω leads to a well-defined time scale separation between the parameter variation and the dynamics of the system while the choice of a_0 and a_1 ensures that a oscillates from values where the unperturbed map of Eq. (1) is chaotic to values where it has a stable fixed point.

A typical sequence of iterates of Eq. (2) (after discarding a transient) is shown in the top panel of Fig. 1. We see an alternation between irregular fluctuations of x and regular episodes due to the oscillating values of a shown in the bottom panel. Due to the slowness of the change of a, the trajectory can relax toward the stable fixed point for those values of a_n where it exists, whereas it follows an irregular trajectory for $a_n \lesssim 3.375$, where the map of Eq. (1) is chaotic.

Since the ϕ dynamics is not mixing, this system clearly has (at least) one invariant measure in its two-dimensional phase space for every ϕ_0, on which ergodic averages are well defined. The Jacobi matrix of Eq. (2) is triangular, which immediately shows that one Lyapunov exponent (corresponding to the ϕ dynamics) is zero, whereas the other $\lambda = \langle \lambda_n \rangle$ is found as an average over an infinitely long trajectory of the instantaneous stretching factors λ_n (middle panel of Fig. 1) defined by

$$\lambda_n = \log \left| \frac{\partial f(a(\phi_n),x_n)}{\partial x_n} \right| .$$

Assuming ergodicity, λ can also be found as an average of the stretching factor over the invariant measure. The seemingly intermittent dynamics shown in the upper panel of Fig. 1 has a well-defined nontrivial Lyapunov exponent, whose sign depends on the details of the x dynamics. The alternation between unstable and stable phases is different from the typical intermittency scenarios [4]. The contributions of the stable and unstable phases to the Lyapunov exponent have a sensitive and subtle dependence on the choice of a_0 and a_1, as we show in Sec. VI.

III. INDISTINGUISHABILITY BETWEEN CHAOTIC AND NONCHAOTIC ORBITS

For the map of Eq. (1), almost any initial condition with $a > a_c$, $|a-a_c| \ll 1$ settles after a transient on a fixed point $\hat{x}(a)$. On the other hand, for $a < a_c$, $|a-a_c| \ll 1$, the invariant set is one-dimensional for a dense set of parameter values, but shows type-I intermittency because of the closeness to the tangent bifurcations. For the map of Eq. (2) during one period of the auxiliary variable ϕ_n, the parameter $a(\phi_n)$ alternates between the periodic and chaotic regime of the map of Eq. (1) and therefore λ has contributions with negative and positive signs. Depending on which contribution has a larger modulus, the overall dynamics is either chaotic or not. This in turn depends on the values of a_0, a_1, and ω.

During the iterates where the trajectory looks regular and is near to $\hat{x}(a)$, the fixed point of the map of Eq. (1), the tangent space dynamics is contracting, and we call the accumulated contraction factor f_c. During the iterations when the trajectory looks irregular, its tangent space dynamics is essentially expanding, and we call the accumulated expansion factor f_e. Then, roughly, the Lyapunov exponent is $\lambda \approx \langle \ln|f_c| + \ln|f_e| \rangle$, where $\langle \cdots \rangle$ now denotes the average over successive periods of ϕ. If we start two trajectories with a distance ϵ at the beginning of the irregular phase, at its end their distance is ϵf_e. This distance will shrink during the regular phase, and at its end will be $\epsilon f_e f_c$. Hence, if $f_e f_c$ on average is smaller than unity, two trajectories will approach each other, and finally will be indistinguishable.

This alternation between contracting and expanding episodes is clearly visible in Fig. 1. A stable periodic orbit thus is as irregular as a chaotic solution, but its irregular segment repeats itself exactly in every period of oscillation of the parameter a, whereas for a trajectory with a positive Lyapunov exponent it does not. Hence, this system can create arbitrarily complicated periodic orbits, since by the choice of ω one can determine the period length and also how many points of the periodic orbit are in the irregular regime. When the ϕ_n dynamics is quasiperiodic instead of periodic (by choosing N as an irrational number), also orbits with negative Lyapunov exponent have nonrepeating irregular segments. By visual inspection, these cannot be distinguished from orbits with a positive Lyapunov exponent.

IV. THE STROBOSCOPIC VIEW

The special dynamics of ϕ_n hinders the two-dimensional map of Eq. (2) from having a fixed point. The shortest periodic orbit can have length N when $\omega=2\pi/N$. Therefore, it makes sense to study the composition of N successive iterates $F(x) = \bigcirc_{n=1}^{N} f(a(\phi_n), x_n)$. When the Lyapunov exponent of Eq. (2) is negative, F should have a stable periodic orbit

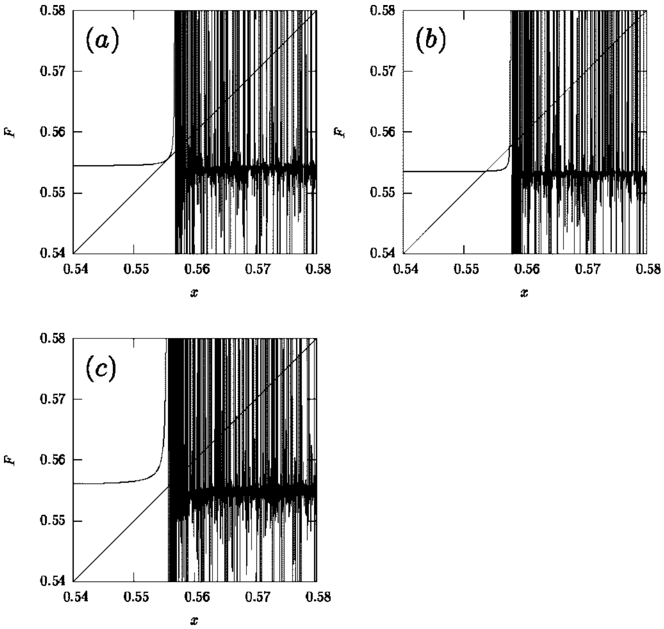

FIG. 2. The graph F as defined in the text. For (a) $a_0 = a_c = 3.375$ there is a tangent bifurcation, for (b) $a_0 = a_c + 0.000\,05$ an almost superstable fixed point, and for (c) $a_0 = a_c - 0.000\,05$ many unstable fixed points. We used $a_1 = 0.1$ and $N = 100$.

or fixed point, whereas it has only unstable periodic orbits and chaotic solutions for positive exponents. In Fig. 2, we show the graph of F for three values of a_0. For $a > a_c$, $a - a_c \ll 1$, $F(x)$ has an almost superstable fixed point. These fixed points are generically born and eliminated by tangent bifurcations, together with their unstable counterparts. For every initial phase ϕ_0, we have a different stroboscopic map $F(x)$. However, they are all topologically conjugate. For our choice of the variation of $a(\phi_n)$, however, the existence or absence of a stable fixed point can best be seen when $\phi_0 \approx 0$, where each orbit of the system Eq. (2) assumes values close to the fixed point of $f(x,a)$ in Eq. (1) for fixed $a \approx a_0 + a_1$.

V. NOISE AND ROUND-OFF EFFECTS

The results described above, and in particular the distinction between motion corresponding to negative and positive Lyapunov exponents, is correct only in the abstract mathematical setting. On a computer, the finite precision of the internal representation of real numbers can enforce the motion onto a complex periodic orbit although its Lyapunov exponent is positive. This happens when the contraction factor f_c during a contracting phase is too strong, so that at the end of this phase the trajectory has no memory of the previous expanding phase. For typical values of a_0 and a_1, we found numerically that orbits with a positive Lyapunov exponent became periodic when the contracting phase contained more than 30 iterates. If these phases are shorter, either because the period N is small enough or because a_0 and a_1 are chosen to be inside the chaotic regime, orbits with positive Lyapunov exponents are nonperiodic, as expected.

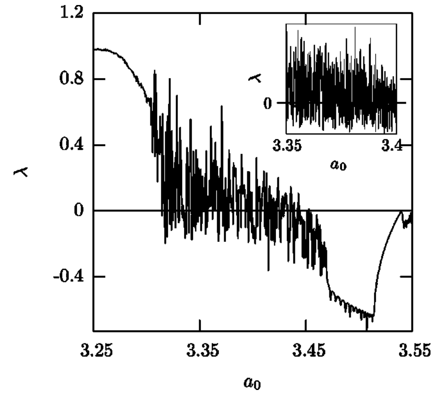

FIG. 3. The Lyapunov exponent λ as a function of the parameter a_0 with $a_1 = 0.1$, $N = 100$. The inset shows the values of λ for $a_0 \in [3.35, 3.4]$.

Hence, with finite precision and large N, one cannot decide, without computation of the Lyapunov exponent, whether the system has a stable periodic orbit or not.

In an experimental realization, instead of computer round-off errors, there is external noise coupled into the system. Equation (1) then has to be modified by adding white noise $\sigma\xi_n$, where $0 < \sigma \ll 1$, $\langle \xi_n \xi_{n'} \rangle = \delta_{n,n'}$, and $\langle \xi_n \rangle = 0$. This has no visible effect on chaotic solutions of Eq. (2), but it does destroy the periodicity of stable periodic solutions. Inside the expanding and hence irregular sections, noise is exponentially amplified and creates orbits which appear chaotic. Systems like Eq. (2) therefore have periodic orbits which are extremely sensitive to external noise, despite linear stability.

VI. PARAMETER DEPENDENCE

The rather complex sequence of tangent bifurcations leading to the creation and destruction of the stable periodic orbits causes the orbits to depend sensitively on the system parameters, so that there is a complicated flipping from periodic to chaotic motion as a function of every single parameter as illustrated in Figs. 3–5. In the first one, we show the Lyapunov exponent λ as a function of a_0 (a_1 and N fixed).

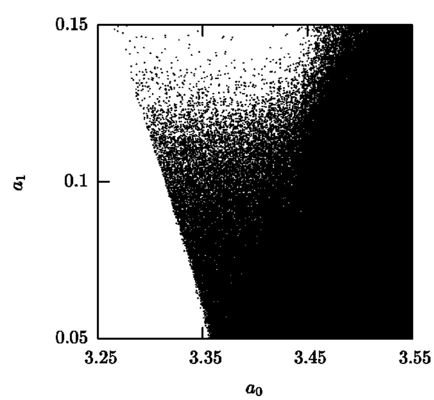

FIG. 4. Lyapunov diagram with $N = 100$. Black dots correspond to a negative Lyapunov exponent, white to a positive one. A total of 65 536 dots are drawn. A cut at $a_1 = 0.1$ corresponds to the previous figure.

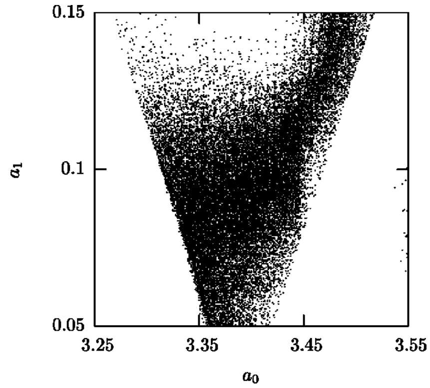

FIG. 5. Black dots represent the ϵ-uncertain points with $\epsilon = 10^{-7}$ and $N=100$.

The rapid change from chaotic to stable periodic solutions leads us to assume that both types of behaviors are supported by a dense set of parameters. The same follows from Fig. 4, where we show the Lyapunov diagram where white (black) dots correspond to a positive (negative) value of λ [5]. There are clearly two regions where λ has a definite sign and an intermediate one where slight changes of a_0 or a_1 have a dramatic effect on the dynamics. We determined the fractal dimension of the boundary between stable and unstable solutions by finding the scaling of the number of ϵ-uncertain points as ϵ is varied [6,7]. A point (a_0, a_1) is ϵ-uncertain if in a neighorhood of radius ϵ there exists at least one point where the Lyapunov exponent has an opposite sign to the one evaluated at (a_0, a_1). In Fig. 5, we show those points of the previous figure that are ϵ-uncertain for $\epsilon = 10^{-7}$. From the scaling of the number of uncertain points with ϵ, we found the box counting dimension D_0 of the boundary between chaotic and stable solutions to be $D_0 \sim 1.985$. This result gives quantitative support to the idea that both stable and unstable orbits have a dense set of parameters. We also found that $D_0 \sim 2$ for $N > 100$ and also for $N < 20$ while it has a minumum value $D_0 \sim 1.8$ for $N \sim 30$.

VII. OSCILLATORY FLOW IN A DUCT

As a first example that displays the behavior discussed previously, we briefly present results of numerical experiments of an oscillatory flow in a duct filled with fluid. This flow can be generated by imposing oscillatory pressure or velocity fields at the ends of the duct with suitably defined phase lags. The stability of these flows can be described in terms of two nondimensional parameters, namely the oscillatory Reynolds number R_δ and the Stokes parameter λ. These are defined by $R_\delta = U\delta/\nu$ and $\lambda = D/\delta$, where U is a characteristic velocity, ν the kinematic viscosity of the fluid, D a characterisic distance of the duct, and δ the Stokes penetration depth. This last quantity is defined by $\delta = \sqrt{\nu/2\omega}$ with ω the frequency of the oscillation. It is a well established fact that zones of distinct dynamical qualitative behavior can be identified in the (R_δ, λ) space. Specifically, it has been experimentally observed that for $\lambda \geqslant 2$ and $R_\delta < 500$, the flow is laminar, while for $R_\delta > 500$, the flow inside the duct is laminar for the phase intervals where the velocity is small while bursts with a frequency much larger than the forcing appear near the end of the acceleration phase [8]. As R_δ is increased, the phase interval where high-frequency oscillations are present gets larger, but it never covers the cycle entirely. The origin of the high-frequency oscillations is the generation of vortices due to the instability of the laminar flow. Numerical results agree with the experiment [9].

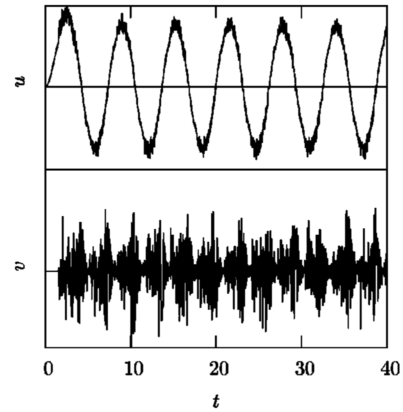

FIG. 6. Velocities u and v as a function of time t in arbitrary units at the center of a two-dimensional duct with expansions at the ends and with an aspect ratio (total length/cross section) of 20, $R_\delta = 1521$, and $\lambda \sim 4$.

In particular, Fig. 6 shows the axial u and transversal v velocities in the middle of a duct. As can be observed, when the velocity is close to zero, the trace is smooth, indicating laminar flow, however high-frequency oscillations appear when the velocity reaches its maximum absolute value in each cycle. Taking $\omega_v = U/D$ as the lower limit of the characteristic frequency of the vortices, the ratio of the vortices frequency to the forcing frequency, ω_v/ω, is $2R_\delta/\lambda$, which for this example is 768. This indicates that the changes in the driving force are slow compared with the internal vortex dynamics corresponding with the conditions discussed in the previous sections.

VIII. DRIPPING FAUCET WITH VARIABLE SUPPLY

Our second example describes the dynamics of a dripping faucet with variable liquid supply. This problem with constant liquid supply has been studied extensively [10]. The time interval T between successive drops shows a complicated bifurcation diagram as the water supply ϵ increases. There are two studies that are important in the context of the present analysis. The model presented by Fuchikami *et al.* [11] is relevant because it is built on sound physical phenomenology, but unfortunately it is not simple, and long time calculations involving many drops are extremely computing-intensive. On the other hand, the model presented by Coullet *et al.* [12], which is based on the former, is useful since it is simple and can be used for exploring the long time behavior. Both models can be adapted to analyze the system when the liquid supply varies as a harmonic function of time and they display a similar behavior with constant and variable liquid supply.

The Coullet model with constant liquid supply ϵ has a complicated bifurcation diagram and we choose a value ϵ_c in such a way that to its left there are period-1 solutions, and to its right a densely chaotic region. We now consider a variable liquid supply that varies harmonically around a value near ϵ_c with a period which is approximately 1000 times larger than the characteristic period of the subsequent drop release with constant liquid supply and with an amplitude $\Delta\epsilon$ chosen in a way that the water supply does not extend outside the period-4 window. These conditions allow the system to visit alternatively a zone of irregular behavior and a zone of period 4 in the map of constant liquid supply. The result of the periodic variation of the water supply is shown in Fig. 7. The continuous sinusoidal line represents the liquid supply and the dots the values of the time intervals between successive drops. As can be clearly seen, the system presents zones of stable and unstable behavior as for the map of Eq. (2).

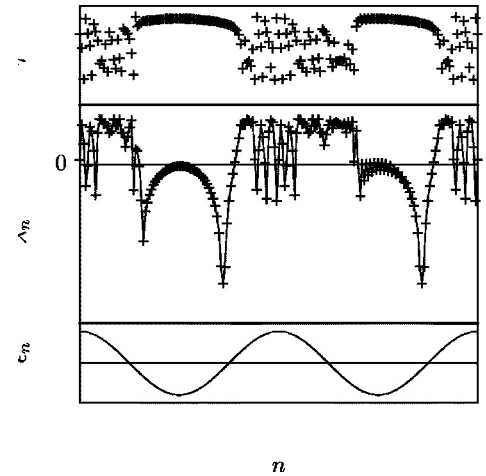

FIG. 7. Two hundred iterates of the map of Coullet *et al.* mentioned in the text (top panel), the instantaneous stretching rates λ_n (middle panel), and ϵ varying harmonically (bottom panel) around $\epsilon_0 = 0.014\,696\,9$ with an amplitude $\epsilon_1 = 0.0004$ and a frequency $\omega = 2\pi/100$.

IX. CONCLUSIONS

We discussed systems with periodic parameter fluctuations which are driven from regular to chaotic motion and back. Although simple in its construction, this type of dynamics creates very complicated orbits with a complex dependence on control parameters. In computer simulations and in real experiments, it is impossible to distinguish the existence of stable complex periodic solutions and of chaotic solutions in the underlying model, since round-off errors and noise interact with the dynamics.

We presented two numerical experiments that illustrate the change of behavior due to a periodic variation of a parameter with a time scale much larger than the natural time scale of the system where the discussion of the impossibility of distinguishing chaos from order is relevant.

Our detailed analysis was based on maps, but all features are found as well in systems with continuous time. It is also not relevant to assume sinusoidal variation of the parameter, so that we expect such a behavior to be rather widespread.

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